JOURNAL OF APPROXIMATION THEORY 54, 196–209 (1988)

Characterization of Weights in Best Rational Weighted Approximation of Piecewise Smooth Functions, II

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Communicated by R. Bojanic

Received April 4, 1986

1. INTRODUCTION

The problem of characterization of weights in weighted L_p rational approximation of piecewise smooth functions f was introduced and studied in [4]. A motivation for the study of this subject is its relationship to the realization of recursive filters. In practice, it is sometimes desirable to include a multiplicative factor s with the rational approximant r_n. This leads to the so-called generalized inverse approximation problem (cf. [1,3]). An example is $f \equiv 1$, and in this case r_n provides an inverse approximation of $1/s$, which is a generalization of the least-squares inverse approximation that guarantees stability [2]. To facilitate our discussion, we need some notation and definitions.

Let

$$\Gamma: 0 = x_0 < x_1 < \cdots < x_{m+1} = 1$$

be a partition of the interval [0,1]. As in [4], we will also use Γ to denote

* This author was supported by the U.S. Army Research Office under Contract No. DAAG 29–84-K-0154.

† This author was a Visiting Scholar at Texas A & M University during the period August–December 1985 and was supported by the National Science Foundation under Grant No. INT-8416057.

0021-9045/88 $3.00

the set $\{x_1, ..., x_m\}$ of interior partition points. Let $U = \{u_1, ..., u_m\}$ be a system of non-negative integers and denote by $A(\Gamma, U)$ the collection of all complex-valued continuous functions f on $[0, 1]$ whose restrictions on each $I_j = [x_j, x_{j+1}]$ are analytic on I_j, $j = 1, ..., m$, and satisfy the joining conditions

$$f^{(s)}(x_j^-) = f^{(s)}(x_j^+), \qquad s = 0, ..., u_j,$$

and

$$f^{(u_j+1)}(x_j^-) \neq f^{(u_j+1)}(x_j^+).$$

If $U = 0 := \{0, ..., 0\}$, then we will simply write $A(\Gamma, U) = A(\Gamma)$. Let w denote an arbitrary weight function, i.e., w is measurable and $0 < w < \infty$ a.e. on $[0, 1]$. For any measurable function f defined on $[0, 1]$, we will use the notation

$$\|f\|_{L_p(w)} = \begin{cases} \left\{\int_0^1 |f(x)|^p\, w(x)\, dx\right\}^{1/p} & \text{if } 0 < p < \infty, \\ \operatorname*{ess\,sup}_{0 \leqslant x \leqslant 1} |f(x)|\, w(x) & \text{if } p = \infty. \end{cases}$$

Of course, if $1 \leqslant p \leqslant \infty$, $\|\cdot\|_{L_p(w)}$ defines a norm for the space $L_p(w)$ of functions f with $\|f\|_{L_p(w)} < \infty$. Let $\mathbf{R}_n[a, b]$ denote the collection of all rational functions p_n/q_n where p_n and q_n are in π_n, the set of all polynomials of degree $\leqslant n$, and are relatively prime, with $q_n(x) \neq 0$ for all x in $[a, b]$. In addition, set $\mathbf{R}_n = \mathbf{R}_n[0, 1]$ and $\mathbf{R} = \bigcup_n \mathbf{R}_n$. Let

$$\Delta: 0 = y_0 < y_1 < \cdots < y_{l+1} = 1$$

be another partition of $[0, 1]$ and $V = \{v_1, ..., v_l\}$ the corresponding system of non-negative integers. Let s be a fixed function in the class $A(\Delta, V)$. The "distance" of f from $s\mathbf{R}_n$ will be denoted by

$$e_n(s, f)_{L_p(w)} := \inf\{\|f - sr_n\|_{L_p(w)} : r_n \in \mathbf{R}_n\},$$

where $0 < p \leqslant \infty$. We also need the following notation introduced in [4]. For any weight function w on $[0, 1]$, set

$$U_p(w) = \left\{x \in [0, 1] : \int_{[x-\delta, x+\delta] \cap [0, 1]} w(t)\, dt = \infty, \text{ for all } \delta > 0\right\}$$

if $0 < p < \infty$ and

$$U_\infty(w) = \left\{x \in [0, 1] : \operatorname*{ess\,sup}_{[x-\delta, x+\delta] \cap [0, 1]} w(x) = \infty, \text{ for all } \delta > 0\right\}$$

For any systems $\Theta = \{\theta_1, ..., \theta_k\}$ and $\mathcal{M} = \{\mu_1, ..., \mu_k\}$ with $0 \leqslant \theta_1 < \cdots \leqslant \theta_k \leqslant 1$ and $\mu_1, ..., \mu_k > 0$, denote by $W_p(\Theta, \mathcal{M})$, $0 < p \leqslant \infty$, the collection of all weight functions w on $[0, 1]$ that satisfy the conditions

$$U_p(w) = \Theta$$

and

$$\prod_{s=1}^{k} |\cdot - \theta_s|^{\mu_s} \in L_p(w).$$

The main result in this paper can be stated as follows.

THEOREM 1. *Let the classes $A(\Gamma, U)$ and $A(\Delta, V)$ be defined as above, s a fixed function in $A(\Delta, V)$, $0 < p \leqslant \infty$, and w a given weight function on $[0, 1]$. Then a necessary and sufficient condition for $e_n(s, f)_{L_p(w)} \to 0$ as $n \to \infty$ where f is an arbitrary function in $A(\Gamma, U) \backslash \mathbf{R}$, is that there exist Θ and $\mathcal{M}$ such that $w \in W_p(\Theta, \mathcal{M})$ and the following conditions are satisfied:*

(i) *The set $\Phi = \{x \in [0, 1]: s(x) = 0\}$ is finite and $\Phi \cap U_p(w) = \phi$; furthermore, if $p = \infty$, then for every $\varphi \in \Phi$*

$$\lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{x \in [\varphi - \delta, \varphi + \delta] \cap [0, 1]} w(x) = 0.$$

(ii) *If $\theta_j = x_{s_1} \in \Gamma$, then*

$$\lim_{\delta \to 0^+} \|\chi_{[\theta_j - \delta, \theta_j]}(\cdot - \theta_j)^{u_{s_1}+1}\|_{L_p(w)} = 0 \tag{1}$$

or

$$\lim_{\delta \to 0^+} \|\chi_{[\theta_j, \theta_j + \delta]}(\cdot - \theta_j)^{u_{s_1}+1}\|_{L_p(w)} = 0. \tag{2}$$

(iii) *If $\theta_j = y_{s_2} \in \Delta$, then*

$$\lim_{\delta \to 0^+} \|\chi_{[\theta_j - \delta, \theta_j]}(\cdot - \theta_j)^{v_{s_2}+1}\|_{L_p(w)} = 0$$

or

$$\lim_{\delta \to 0^+} \|\chi_{[\theta_j, \theta_j + \delta]}(\cdot - \theta_j)^{v_{s_2}+1}\|_{L_p(w)} = 0.$$

Here and throughout, χ_J denotes, as usual, the characteristic function of the set J.

2. Proof of the Necessity Condition

Suppose that

$$e_n(s, f)_{L_p(w)} \to 0 \tag{3}$$

as $n \to \infty$, for any $f \in A(\Gamma, U) \backslash \mathbf{R}$. The proof of the existence of Θ and $\mathscr{M}$ such that $w \in W_p(\Theta, \mathscr{M})$ is similar to that given in [4] (of course, it is possible that both of Θ and $\mathscr{M}$ are empty). Next we prove the necessity of the conditions in (i). Assume that Φ is an infinite set. Then s vanishes identically on some interval $I_j = [x_j, x_{j+1}]$. Let $f \in A(\Gamma, U) \backslash \mathbf{R}$ with $f(x) = 1$ for $x \in I_j$. By (3), there exists a sequence $\{r_n\} \subset \mathbf{R}$ such that

$$\|\chi_{I_j}\|_{L_p(w)} \leqslant \|f - s\, r_n\|_{L_p(w)} \to 0.$$

It follows that $w(x) = 0$ for almost all x on I_j, which is a contradiction to our assumption on w. Set $\Phi = \{\varphi_1, ..., \varphi_q\}$ and let $\{r_n\}$ be a sequence in $\mathbf{R}$ such that

$$\rho_n := \| f_0 - s\, r_n \|_{L_p(w)} \to 0,$$

where $f_0 \in A(\Gamma, U)$ and satisfies $f_0(x) > 1$ for all $x \in [0, 1]$. For every fixed n, there exists a positive δ_n such that

$$\max_j \sup_x |\chi_{[\varphi - \delta_n, \varphi_j + \delta_n] \cap [0,1]}(x)\, s(x)\, r_n(x)| \leqslant 1/2.$$

Thus we obtain

$$\sum_{j=1}^{q} \|\chi_{[\varphi_j - \delta_n, \varphi_j + \delta_n] \cap [0,1]}\|_{L_p(w)} \leqslant 2q\rho_n \to 0$$

as $n \to \infty$. It is easy to see that this is equivalent to the conditions in (i).

Now we will show that conditions (ii) and (iii) are also necessary.

Let us first assume that $\theta_j = x_{s_1} \in \Gamma \backslash \Delta$. Since $\Phi \cap U_p(w) = \phi$, there exists a small positive δ^* such that s does not vanish on $[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]$, and without loss of generality, we may assume that on this set $s > \varepsilon^* > 0$, and since $\theta_j \notin \Delta$, that s is analytic there. By [4], there exist $r_n \in \mathbf{R}$ such that

$$\rho_n^* := \left\| \left| \frac{f}{s} - r_n \right| \chi_{[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0,1]} \right\|_{L_p(w)} \to 0, \tag{4}$$

where

$$f(x) = (x - \theta_j)_+^{u_{s_j} + 1}, \text{ for } x \in [\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1].$$

If both (1) and (2) do not hold, then r_n must be of the form

$$r_n(x) = \frac{(x-\theta_j)_+^{u_{s_1}+1}\, p_{n-u_{s_1}-1}(x)}{q_n(x)},$$

where $p_{n-u_{s_1}-1} \in \pi_{n-(u_{s_1}+1)}$ and $q_n \in \pi_n$. Hence, by (4)

$$\rho_n^* = \Bigg\| \left| \frac{\operatorname{sgn}(\cdot - \theta_j)+1}{2s(\cdot)} - \frac{p_{n-u_{s_1}-1}(\cdot)}{q_n(\cdot)} \right| \times (\cdot - \theta_j)^{u_{s_1}+1} \chi_{[\theta_j-\delta^*,\, \theta_j+\delta^*] \cap [0,1]}(\cdot) \Bigg\|_{L_p(w)} \to 0$$

and it follows that

$$\frac{d}{dx} \frac{p_{n-u_{s_1}-1}(x)}{q_n(x)} \bigg|_{x=\theta_j} = \infty$$

(cf. the proof of Theorem 1 in [4]). But this is impossible. Similarly, if $\theta_j = y_{s_2} \in \Delta \backslash \Gamma$, then we arrive at a similar contradiction when we assume that both (1) and (2) do not hold.

Now suppose that $\theta_j = x_{s_1} = y_{s_2} \in \Gamma \cap \Delta$, and set $v = \min(u_{s_1}, v_{s_2})$. Assume that both

$$\lim_{\delta \to 0^+} \|\chi_{[\theta_j-\delta,\, \theta_j]}(\cdot)(\cdot - \theta_j)^{v+1}\|_{L_p(w)=0} \tag{5}$$

and

$$\lim_{\delta \to 0^+} \|\chi_{[\theta_j,\, \theta_j+\delta]}(\cdot)\, (\cdot - \theta_j)^{v+1}\|_{L_p(w)=0} \tag{6}$$

do not hold. Then since $\Phi \cap U_p(w) = \phi$, we may assume that $s(x) > \varepsilon^* > 0$ on some small interval $[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]$. From (3), we obtain (4) for some $\{r_n\} \subset \mathbf{R}$. Set

$$f^*(x) = \frac{f(x)}{s(x)}, \qquad x \in [\theta_j - \delta^*, \theta_j + \delta^*] \cap [0,\ 1].$$

Then both of the restrictions of f^* on $[\theta_j - \delta^*, \theta_j]$ and $[\theta_j, \theta_j + \delta^*]$ are analytic on the corresponding intervals. Furthermore, $(d^s/dx^s)\ f^*(x)$, $s = 0, ..., v$, are continuous at $x = \theta_j$ and

$$\frac{d^{v+1}}{dx^{v+1}} f^*(\theta_j^-) \neq \frac{d^{v+1}}{dx^{v+1}} f^*(\theta_j^+).$$

Set

$$p^*(x) = \sum_{s=0}^{\nu} \frac{1}{s!} \left(\frac{d^s}{dx^s} f^*(\theta_j) \right) (x-\theta_j)^s.$$

Then $f^* - p^*$ is of the form

$$f^*(x) - p^*(x) = g^*(x)(x-\theta_j)^{\nu+1} \qquad \text{where } x \in [\theta_j - \delta^*, \theta_j + \delta^*] \cap [0, 1]$$

and g^* satisfies the inequality

$$g^*(\theta_j^-) \neq g^*(\theta_j^+).$$

If both (5) and (6) do not hold, then r_n must be of the form

$$r_n(x) = p^*(x) + \frac{(x-\theta_j)^{\nu+1} p_{n-(\nu+1)}(x)}{q_n(x)}.$$

From (4) it follows that

$$\left\| \left| g^*(\cdot) - \frac{p_{n-(\nu+1)}(\cdot)}{q_n(\cdot)} \right| (\cdot - \theta_j)^{\nu+1} \chi_{[\theta_j - \delta^*, \theta_j + \delta^*] \cap [0,1]}(\cdot) \right\|_{L_p(w)} \to 0,$$

yielding

$$\frac{d}{dx} \frac{p_{n-(\nu+1)}(x)}{q_n(x)} \bigg|_{x=\theta_j} = \infty,$$

which is again a contradiction.

3. Proof of the Sufficiency Condition

In order to prove that the conditions in Theorem 1 are sufficient we need several lemmas. The first one was established in [4].

Lemma 1. *Let* $\eta = \exp(-1/\sqrt{n})$, $\xi_1, ..., \xi_q \in [-1, 0) \cup (0, 1]$, $\mu > 0$, *and* $\mu_j > 0$, $j = 1, ..., q$. *Then for any constants* δ, B, C, ε, *and* $\varepsilon_1, ..., \varepsilon_q$ *satisfying*

$$0 < \delta < 1/2, \qquad 1 < B^{[\mu]+1} < e, \qquad c > 1, \qquad \text{and} \quad \varepsilon > 0,$$

there exist rational functions $r_n \in R_{m_n}[-1, 1]$ *with* $m_n = n + O(\sqrt{n})$ *such that*

$$|\operatorname{sgn} x - r_n(x))$$

$$= \begin{cases} O(1) & \text{for} \quad x \in [-\eta^n, \eta^n], \\ O\left(\left(\frac{B^{[\mu]+1}}{e}\right)^{\sqrt{n}}\right) \prod_{\xi_j > 0} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} |x - \varepsilon B^{-\sqrt{n}}|^{\mu} & \\ & \text{for} \quad x \in [\eta^n, 1], \\ O\left(\left(\frac{B^{[\mu]+1}}{e}\right)^{\sqrt{n}}\right) \prod_{\xi_j < 0} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} |x - \varepsilon B^{-\sqrt{n}}|^{\mu} & \\ & \text{for} \quad x \in [-1, -\eta^n], \\ O(C^{-\sqrt{n}}) \prod_{j=1}^{q} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} & \text{for} \quad \delta \leqslant |x| \leqslant 1, \end{cases}$$

where the "O" terms are independent of x.

The second lemma we need is a well known result of Bernstein.

LEMMA 2. *Let f be analytic on $[a, b]$. Then there exists a sequence of polynomials p_n in π_n and a positive λ such that*

$$\max_{a \leqslant n \leqslant b} |f(x) - p_n(x)| = O(e^{-\lambda n}).$$

LEMMA 3. *Let $\delta, \beta > 0$ be given, $\Gamma = \{x_1, x_2\} \subset (0, 1)$, $U = \{u_1, u_2\}$ and $w \in W_p(\Theta, \mathcal{M})$, $0 < p \leqslant \infty$, for some $\Theta = \{\theta_1, ..., \theta_k\}$ and $\mathcal{M} = \{\mu_1, ..., \mu_k\}$. Suppose that f is a piecewise analytic function of the form*

$$f(x) = (x - x_1)^{u_1 + 1} (x - x_2)^{u_2 + 1} g(x) \chi_{[x_1, x_2]}(x),$$

where g is analytic on the interval $I_1 = [x_1, x_2]$. Then there exists $r_n \in R_n$, $n \geqslant n_0$ such that

$$\|f - r_n\|_{L_p(w)} = O(A^{-\sqrt{n}} + \mathscr{E}_n(B)) \tag{7}$$

for some $A > 1$ and $B > 1$, where

$$\mathscr{E}_n(B) = \sum_{\theta_s = x_j \in \Gamma} \min(\mathscr{E}_{n,s}^{-}(B), \mathscr{E}_{n,s}^{+}(B))$$

with

$$\mathscr{E}_{n,s}^{-}(B) = (B^{-\sqrt{n}})^{u_j + 1} \|\chi_{[\theta_s - \delta, \theta_s - B^{-\sqrt{n}}]}\|_{L_p(w)} + \|\chi_{[\theta_s - B^{-\sqrt{n}}, \theta_s]}(\cdot)(\cdot - \theta_s)^{u_j + 1}\|_{L_p(w)}$$

and

$$\mathscr{E}^+_{n,s}(B) = (B^{-\sqrt{n}})^{\mu_j+1} \|\chi_{[\theta_s + B^{-\sqrt{n}},\, \theta_s + \delta]}\|_{L_p(w)} + \|\chi_{[\theta_s,\, \theta_s + B^{-\sqrt{n}}]}(\cdot)(\cdot - \theta_s)^{\mu_j+1}\|_{L_p(w)}.$$

Furthermore, $\{r_n\}$ *converges uniformly to* f *on* $[0, 1]$.

The proof of this lemma is similar to that of Lemma 4 in [4]. We assume, without loss of generality, that $\delta > 0$ is so small that g is analytic on $[x_1 - \delta, x_2 + \delta]$ and $\theta_s \notin [x_1 - \delta, x_1) \cup (x_2, x_2 + \delta]$, $s = 1, ..., k$. Construct a polynomial p_0 of degree $\leqslant \sum_{\theta_s \in [x_1, x_2]} ([\mu_s] + 1)$ such that

$$p_0(x) - g(x) = \prod_{\theta_s \in [x_1, x_2]} (x - \theta_s)^{[\mu_s]+1} \tilde{g}(x),$$

where $\tilde{g}$ is also analytic on $[x_1 - \delta,\ x_2 + \delta]$. By Lemma 2, there is a polynomial p_1 of degree $\leqslant K[\sqrt{n}] - \sum_s ([\mu_s] + 1)$ such that

$$|\tilde{g}(x) - p_1(x)| = O(e^{-\sqrt{n}})$$

uniformly for $x \in [x_1 - \delta, x_2 + \delta]$. Set

$$p_2(x) = p_0(x) - \prod_{\theta_s \in [x_1, x_2]} (x - \theta_s)^{[\mu_s]+1} p_1(x).$$

Then p_2 is a polynomial of degree $K[\sqrt{n}]$ and

$$p_2(x) - g(x) = O(e^{-\sqrt{n}}) \prod_{\theta_s \in [x_1, x_2]} |x - \theta_s|^{\mu_s} \tag{8}$$

uniformly for $x \in [x_1 - \delta, x_2 + \delta]$.

There are the following possible cases:

(1) $\Theta \cap \Delta = \phi$;

(2) $x_1 = \theta_{s_0} \in \Theta$, $x_2 \notin \Theta$;

(3) $x_1 \notin \Theta$, $x_2 = \theta_{s_0} \in \Theta$; or

(4) both $x_1 = \theta_{s_1}$ and $x_2 = \theta_{s_2}$ belong to the set Θ.

For simplicity, we will only give the proof for the case (4) since the others can be verified similarily. Set

$$x' = x_1 - 2B^{-\sqrt{n}} \qquad \text{and} \qquad x'' = x_2 - 2B^{-\sqrt{n}},$$

where

$$B = \frac{1}{2} \min \left\{ 1 + \exp\left(\frac{1}{[\mu_{s_1}] + 1}\right),\ 1 + \exp\left(\frac{1}{[\mu_{s_2}] + 1}\right) \right\}.$$

If n is sufficiently large, then we have

$$\theta_s \notin [x', x_1) \cup [x'', x_2), \qquad s=1, ..., k.$$

By (8), we see that

$$\begin{aligned}
&\chi_{[x', x'']}(x)(x-x_1)^{u_1+1}(x-x_2)^{u_2+1} p_2(x) - f(x) \\
&\quad = O(e^{-\sqrt{n}}) \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} - \chi_{[x', x_1]}(x)(x-x_1)^{u_1+1}(x-x_2)^{u_2+1} p_2(x) \\
&\qquad + \chi_{[x'', x_2]}(x)(x-x_1)^{u_1+1}(x-x_2)^{u_2+1} p_2(x). \qquad (9)
\end{aligned}$$

Write

$$\chi_{[x', x'']}(x) = \tfrac{1}{2}\{\operatorname{sgn}(x-x') - \operatorname{sgn}(x-x'')\}.$$

By Lemma 3, there are rational functions $\tilde{r}$ and $\hat{r}$ of degree $n+O(\sqrt{n})$ such that

$$|\tilde{r}(x) - \operatorname{sgn}(x-x')| = \begin{cases} O(1) & \text{for } x \in [0, 1], \\ O(B^{[\mu_{s_1}]+1}/e)^{\sqrt{n}} \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} & \text{for } |x-x'| \geqslant \eta^n \text{ and } x \in [0, 1], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} & \text{for } |x-x'| \geqslant \delta \text{ and } x \in [0, 1], \end{cases}$$

and

$$|\hat{r}(x) - \operatorname{sgn}(x-x'')| = \begin{cases} O(1) & \text{for } |x-x''| \leqslant \eta^n, \\ O(B^{[\mu_{s_2}]+1}/e)^{\sqrt{n}} \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} & \text{for } |x-x''| > \eta^n \text{ and } x \in [0, 1], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} & \text{for } |x-x''| \geqslant \delta \text{ and } x \in [0,-1], \end{cases}$$

where C is an arbitrarily given positive constant. It is known that

$$p_2(x) = O(e^{\lambda' \sqrt{n}})$$

for some $\lambda' > 0$ uniformly for $x \in [0, 1]$. Set $C = \exp(\lambda' + 1)$ and

$$r^*(x) = \frac{1}{2}(\tilde{r}(x) - \hat{r}(x))(x-x_1)^{u_1+1}(x-x_2)^{u_2+1} p_2(x).$$

Then we obtain, using (9),

$$\|f-r^*\|_{L_p(w)} = O(e^{-\sqrt{n}}) + O(B^{[\mu_{s_1}]+1}e)^{\sqrt{n}} + O(B^{[\mu_{s_2}]+1}/e)^{\sqrt{n}} + O(\mathscr{E}^-_{n,s_1}(B)) + O(\mathscr{E}^-_{n,s_2}(B)),$$

and this, in turn, assures the existence of $r_n \in \mathbf{R}_n$ such that

$$\|f-r_n\|_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}^-_{n,s_1}(B) + \mathscr{E}^-_{n,s_2}(B))$$

for some $A>1$ and $B>1$. Similarly, there exist $r_n \in R_n$ such that

$$\|f-r_n\|_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}^-_{n,s_1(B)} + \mathscr{E}^+_{n,s_2}(B)),$$
$$\|f-r_n\|_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}^+_{n,s_1}(B) + \mathscr{E}^-_{n,s_2}(B)),$$

or

$$\|f-r_n\|_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}^+_{n,s_1}(B) + \mathscr{E}^+_{n,s_2}(B)).$$

Hence, combining these estimates, we obtain (7). By the same proof, we can also conclude that $\{r_n\}$ converges uniformly to f on $[0, 1]$. This completes the proof of the lemma.

Remark 1. *A similar proof also shows that the result in Lemma* 3 *also holds for* $x_1=0$ *and/or* $x_2=1$.

We are now ready to prove that the conditions in Theorem 1 are sufficient.

Let $\Phi=\{\varphi_1, ..., \varphi_q\}$ where $\varphi_1 < \cdots < \varphi_q$. We will only consider the case where $\varphi_1 \neq 0$ and $\varphi_q \neq 1$, since the other cases can be verified in a similar manner. Choose small $\delta_j^{(1)}$ and $\delta_j^{(2)}, j=1, ..., q$, such that

$$|s(\varphi_j-\varphi_j^{(1)})| = |s(\varphi_j+\delta_j^{(2)})| := h_j,$$
$$|s(x)| \leqslant h_j \qquad \text{for } x \in [\varphi_j-\delta_j^{(1)}, \varphi_j+\delta_j^{(2)}],$$

$j=1, ..., q$, and $U_p(w) \cap Z = \phi$, where

$$Z = \bigcup_j [\varphi_j-\delta_j^{(1)}, \varphi_j+\delta_j^{(2)}].$$

Now,

$$\|f-sr_n\|_{L_p(w)} \leqslant \left\| s\left|\frac{f}{s}-r_n\right| \chi_{[0,1]\setminus Z}\right\|_{L_p(w)} + \| |f-s\,r_n|\,\chi_Z\|_{L_p(w)} := H_1+H_2.$$

Define a continuous function g on $[0, 1]$ as follows:

$$g(x)=\begin{cases} \dfrac{f(x)}{s(x)} & \text{for} \quad x\in[0,1]\backslash Z, \\ \text{linear} & \text{otherwise.} \end{cases}$$

By Lemma 3, it is easy to show that there exist $r_n \in R_n$, $n \geqslant n_0$, such that

$$H_1 \leqslant \|s\|_\infty \; \|g-r_n\|_{L_p(w)} \to 0, \tag{10}$$

as $n \to \infty$, $\{r_n\}$ converges to g uniformly on Z, and (7) holds. Hence, for all large n, we have

$$\sup_{x\in[\varphi_j-\delta_j^{(1)},\, \varphi_j+\delta_j^{(2)}]} |r_n(x)| \leqslant 2\, \|f\|_\infty\, h_j^{-1}, \qquad j=1, ..., q.$$

It follows that

$$H_2 \leqslant \|f\|_\infty\, \|\chi_Z\|_{L_p(w)} + 2\, \|f\|_\infty \sum_{j=1}^{q} h_j^{-1}\, \|s\, \chi_{[\varphi_j-\delta_j^{(1)},\, \varphi_j+\delta_j^{(2)}]}\|_{L_p(w)}. \tag{11}$$

According to the assumption (i) of the theorem, for any given $\varepsilon > 0$, we can choose $\delta_j^{(1)} > 0$ and $\delta_j^{(2)} > 0$, $j=1, ..., q$, such that

$$\|\chi_Z\|_{L_p(w)} < \varepsilon.$$

Hence, we obtain

$$H_2 \leqslant C_q\, \|f\|_\infty\, \varepsilon \tag{12}$$

for some constant C_q depending only on q. Combining (10), (11), and (12), we arrive at

$$e_n(s,f)_{L_p(w)} \to 0,$$

as $n \to \infty$. This completes the proof of the theorem.

4. Approximation Order

We will establish the following result.

THEOREM 2. *Let the classes* $A(\Gamma, U)$ *and* $A(\Delta, V)$ *be given as above,* $0<p\leqslant\infty$, *and s and w satisfy the conditions in Theorem* 1. *If* $s(x)>0$ *for all* $x\in[0,1]$, *then*

(i) *there exist* $A>1$ *and* $B>1$ *such that for every* f *in* $A(\Gamma, U)$

$$e_n(s,f)_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}_n(B)),$$

(ii) *there is a* $\lambda>0$ *such that for every* f *in* $A(\Gamma, U)$

$$e_n(s,f)_{L_p(w)} = O(e^{-\lambda\sqrt{n}})$$

whenever

$$\max_{\theta_j\in\Gamma\cup\Delta} \mu_j < \min_{1\leqslant s\leqslant m,\ 1\leqslant s'\leqslant l} \{u_s+1,\ v_{s'}+1\}. \tag{13}$$

Here,

$$\mathscr{E}_n(B) = \sum_{\theta_s\in\Gamma\cup\Delta} \min(\mathscr{E}^-_{n,s}(B),\ \mathscr{E}^+_{n,s}(B))$$

with $\mathscr{E}^-_{n,s}(B)$ *and* $\mathscr{E}^+_{n,s}(B)$ *defined, similar to the notations used in Lemma* 3, *as follows:*

(1) *If* $\theta_s = x_j \in \Gamma\backslash\Delta$, *then*

$$\begin{aligned}\mathscr{E}^-_{n,s}(B) &= (B^{-\sqrt{n}})^{u_j+1}\, \|\chi_{[x_j-\delta,\, x_j-B^{-\sqrt{n}}]}\|_{L_p(w)} \\ &\quad + \|\chi_{[x_j-B^{-\sqrt{n}},\, x_j]}(\cdot)(\cdot-x_j)^{u_j+1}\|_{L_p(w)},\\ \mathscr{E}^+_{n,s}(B) &= (B^{-\sqrt{n}})^{u_j+1}\, \|\chi_{[x_j+B^{-\sqrt{n}},\, x_j+\delta]}\|_{L_p(W)} \\ &\quad + \|\chi_{[x_j,\, x_j+B^{-\sqrt{n}}]}(\cdot)(\cdot-x_j)^{u_j+1}\|_{L_p(w)}.\end{aligned}$$

(2) *If* $\theta_s = y_j \in \Delta\backslash\Gamma$, *then* $\mathscr{E}^-_{n,s}(B)$ *and* $\mathscr{E}^+_{n,s}(B)$ *are defined as above with the exception that* x_j *and* u_j *are replaced by* y_j *and* v_j, *respectively.*

(3) *If* $\theta_s = x_{j_1} = y_{j_2} \in \Delta\cap\Gamma$, *then*

$$\begin{aligned}\mathscr{E}^-_{n,s}(B) &= (B^{-\sqrt{n}})^{\min(u_{j_1},\, v_{j_2})+1}\, \|\chi_{[\theta_s-\delta,\, \theta_s-B^{-\sqrt{n}}]}\|_{L_p(w)} \\ &\quad + \|\chi_{[\theta_s-B^{-\sqrt{n}},\, \theta_s]}(\cdot)\cdot-\theta_s)^{\min(u_{j_1},\, v_{j_2})+1}\|_{L_p(w)}\end{aligned}$$

and $\mathscr{E}^+_{n,s}(B)$ *is defined similarly.*

We now sketch a proof of this result. If $s>0$, then it follows that

$$\|f-s\ r_n\|_{L_p(w)} \leqslant \|s\|_\infty\ \|g-r_n\|_{L_p}(w), \tag{14}$$

where $g=f/s\in A(\Gamma^*, U^*)$, $\Gamma^*=\Gamma\cup\Delta=\{z_1, ..., z_{m'}\}$ with $z_1<z_2<\cdots<z_{m'}$, $U^*=\{u_1^*, ..., u_{m'}^*\}$, and $u_j^*, j=1, ..., m'$, defined by

$$u_j^* = \begin{cases} u_j & \text{if } z_j\in\Gamma\backslash\Delta,\\ v_j & \text{if } z_j\in\Delta\backslash\Gamma,\\ \min(u_j, v_j) & \text{if } z_j\in\Delta\cap\Gamma.\end{cases}$$

Hence, there is a polynomial p_0 of degree $\leqslant \sum_{j=1}^{m'} u_j^* + m'$ such that

$$g(x) - p_0(x) = \sum_{j=0}^{m'} (x-x_j)^{u_j^*+1} (x-x_{j+1})^{u_{j+1}^*+1} g_j(x) \chi_{I_j}(x)$$

$$= \sum_{j=0}^{m'} f_j(x) \tag{15}$$

say, where $I_j = [z_j, z_{j+1}]$, $j=0, ..., m'$, and g_j is analytic on I_j. By Lemma 3 and Remark 1, we see that there exist $A_j > 1$ and $B_j > 1$, $j = 0,..., m'$, such that

$$e_n(f_j)_{L_p(w)} = O(A_j^{-\sqrt{n}}) + O(\mathscr{E}_n(B_j)), j = 0, 1, ..., m'. \tag{16}$$

Then (14), (15), and (16) together give the conclusion (i) in Theorem 2.
If the condition (13) is satisfied, then it is easy to see that

$$\mathscr{E}_n(B_j) = O(\bar{B}^{-\sqrt{n}})$$

for some $\bar{B}_j > 1$. Thus, (ii) follows from (i).

Remark 2. *Condition* (13) *cannot be deleted. In fact if there is a* $\theta_{j_0} \in \Delta \cap \Gamma$ *such that* $\mu_{j_0} = \min\{u_s + 1, v_s + 1\}$, *then conclusion* (ii) *of Theorem* 2 *does not hold.*

5. Approximation of Piecewise Smooth Functions

We need some notation. Suppose that Γ and U are given as in Section 1 and q is a positive integer such that $\max_j u_j < q$. Denote by $C^q(\Gamma, U)$ the collection of all complex-valued continuous functions f on $[0, 1]$ whose restrictions on each $I_j = [x_j, x_{j+1}]$ belong to $C^q(I_j)$, the class of functions with qth order continuous derivatives on I_j, and satisfy the joining conditions

$$f^{(s)}(x_j^-) = f^{(s)}(x_j^+), \qquad s = 0, ..., u_j,$$

with

$$f^{(u_j+1)}(x_j^-) \neq f^{(u_j+1)}(x_j^+)$$

for $j = 0, ..., m$.
By modifying the proofs in the above discussions, we have also established analogous results for the class $C^q(\Gamma, U)$. We state these results without proof.

THEOREM 3. *Let s be a given function in* $A(\Delta, V)$, $0 < p \leqslant \infty$, *and w a given weight function on* $[0, 1]$. *Then a necessary and sufficient condition for* $e_n(s, f)_{L_p(w)} \to 0$ *as* $n \to \infty$, *where f is an arbitrary function in* $C^q(\Gamma, U)$, *is that the conditions of Theorem* 1 *are satisfied and* $\mu_j \leqslant q$ *for all* $j = 1, ..., k$.

THEOREM 4. *Let s and w satisfy the conditions in Theorem* 3 *with* $0 < p \leqslant \infty$. *If* $s(x) > 0$ *for all* $x \in [0, 1]$, *then*

(i) *there exists* $B > 1$ *such that for every f in* $C^q[\Gamma, U)$

$$e_n(s, f)_{L_p(w)} = O(\mathscr{E}_n(B)) + O\left(\frac{1}{n}\sum_{j=0}^{m} \omega\left(f_j, \frac{1}{n}\right)_{L_p}\right),$$

where f_j *denotes the restriction of f on* I_j *and* $\omega(f_j, 1/n)_{L_p}$ *the* L_p*-modulus of continuity of* f_j, *and*

(ii)

$$e_n(s, f)_{L_p(w)} = O\left(\frac{1}{n^q}\sum_{j=0}^{m} \omega\left(f_j, \frac{1}{n}\right)_{L_p}\right)$$

provided

$$\max_{\theta_j \in \Gamma \cup \Delta} \mu_j < \min_{1 \leqslant s \leqslant m,\ 1 \leqslant s' \leqslant l} \{u_s + 1, v_{s'} + 1\}.$$

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